NAME:

UW ID: | | | | | | | |

Academic Honesty Statement: All work on this exam is my own.

Signature:

INSTRUCTIONS

1. This exam contains eight (8) printed pages. Check that on your exam the bottom of the last page says **END OF EXAM**.
2. The points for each question are indicated at the beginning of each question.
3. **Show all your work**, unless the problem says otherwise explicitly. An answer without work shown will receive little or no credit.
4. If you need more space for your answer, use the back of the page and indicate that you have done so.
5. You may use one sheet of handwritten notes on a 8.5×11 inch paper, both sides. You may use a TI-30X IIS calculator. No other resources are allowed.
6. Raise your hand if you have a question.
7. Time allowed: 50 minutes.

1	/15
2	/20
3	/10
4	/20
5	/10
6	/15
Total	/90

(1) (i) [10 points] Compute the inverse matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, showing all row operations.

(ii) [5 points] Solve the following linear system, using part (i). Show all your work.

$$\begin{cases} x + y + z = 0 \\ \phantom{x + {}} y + z = 1 \\ 2x \phantom{{}+y} + z = 1. \end{cases}$$

(2) Suppose the following matrices A and B are equivalent, i.e. we can get from A to B by applying row operations. Note that B is in echelon form.

$$A = \begin{bmatrix} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \vec{\mathbf{a}}_3 & \vec{\mathbf{a}}_4 & \vec{\mathbf{a}}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

(a) [5 points] Find a basis for row(A), if possible using the given information. If not enough information is given, write "not possible". No justification is needed.

(b) [5 points] Is $\{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_3, \vec{\mathbf{a}}_5 - \vec{\mathbf{a}}_4\}$ a basis for col(A)? Explain why or why not.

(c) [8 points] Find a basis for null(A). Show all your work.

(d) [2 points] Consider the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$. ($A$ is defined on the previous page.) Is T one-to-one? No justification needed.

(3) (a) [5 points] Let $A = \begin{bmatrix} 8 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$. Find a matrix B such that $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, showing all your work, or explain why this is not possible.

(b) [5 points] Which of the following statements must be true for any matrices A and B? Assume the matrix product AB is well-defined. Circle all that apply. No justification needed.

$\mathrm{col}(A) \subset \mathrm{col}(AB)$	$\mathrm{null}(A) \subset \mathrm{null}(AB)$	$\mathrm{col}(A) \subset \mathrm{col}(2A)$
$\mathrm{col}(AB) \subset \mathrm{col}(A)$	$\mathrm{null}(B) \subset \mathrm{null}(AB)$	$\mathrm{null}(A) \subset \mathrm{null}(2A)$

(4) Suppose the matrix A describes a $180°$ rotation in $\mathbb{R}^3$, around some line L through the origin. In other words, for every $\vec{\mathbf{v}} \in \mathbb{R}^3$,

$$A\vec{\mathbf{v}} = \text{rotation of } \vec{\mathbf{v}} \text{ around } L \text{ by } 180°.$$

(a) [5 points] What is the nullity of $I - A$? Explain your answer with geometric reasoning.

(b) [10 points] Suppose $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. which describes a $180°$ rotation around some line in $\mathbb{R}^3$. Let S be the subset of vectors in $\mathbb{R}^3$ defined by

$$S = \{\vec{\mathbf{v}} : A\vec{\mathbf{x}} = \vec{\mathbf{v}} \quad \text{or} \quad A\vec{\mathbf{v}} = -\vec{\mathbf{v}}\}.$$

Is S a subspace of $\mathbb{R}^3$? Explain your answer. (Hint: make a sketch of S.)

(c) [5 points] Suppose $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Find a nonzero vector $\vec{\mathbf{v}}$ such that $A\vec{\mathbf{v}} = \vec{\mathbf{v}}$.

Show all your work.

(5) [10 points] Suppose $A = \begin{bmatrix} 1 & 1 & x \\ 1 & x & 1 \\ x & 1 & 1 \end{bmatrix}$. Find all values of x and y such that

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & y \\ 1 & y & 1 \\ y & 1 & 1 \end{bmatrix}.$$

If no values are possible, write "not possible". Show all your work, and put a **box around your answer**.

(6) [5 points each] In each of the following, either give an example or write "not possible". No justification is necessary.

(a) A 2×3 matrix B such that $\operatorname{col}(B) = \operatorname{col}\left(\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}\right)$.

(b) A 3×3 matrix A such that $\operatorname{rank}(A) = 2$ and $A^2 = 0$.

(c) Two square matrices A, B such that $A + B = I$ and $AB \neq BA$.

(7) [EXTRA CREDIT] Consider the following claim and proof regarding square matrices A and B:

Claim: If $A^2 = B^2$ then $A = B$ or $A = -B$.

Proof: (We use 0 here to mean the zero matrix.)

(i) If $A^2 = B^2$, then $A^2 - B^2 = 0$.

(ii) For any same-size square matrices A and B, $A^2 - B^2 = (A - B)(A + B)$.

(iii) If $(A - B)(A + B) = 0$, then $A - B = 0$ or $A + B = 0$.

(a) [2 points] Which of the above steps is incorrect? Circle all that apply. No justification is needed.

(i) (ii) (iii)

(b) [2 points] Find an example of square matrices A and B such that

$$A^2 = B^2 \qquad \text{but} \quad A \neq B \quad \text{and} \quad A \neq -B.$$

(Hint: pick a "nice" matrix for B.)

END OF EXAM